THE UNIVERSITY OF NORTHERN COLORADO

The Number of Subgroups of the Dihedral Group D(n)

MATH 599

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Abstract

This research paper determines a formula for the number of subgroups of the dihedral groups D(n). I created the operation tables and lattice of subgroups for D(3) through D(8). After creating the lattice of subgroups I determined the elements of D(n) that generate each subgroup of D(n). This led to the formula $S_n = \tau(n) + \sigma(n)$, where S_n represents the number of subgroups of D(n), $\tau(n)$ represents the number of positive divisors of n, and $\sigma(n)$ represents the sum of the positive divisors of n. Students in a secondary classroom will be asked to investigate the symmetries of various figures, the permutation of the figures vertices, and the composition of those permutations.

Keywords: {Dihedral Group D(n), $\tau(n), \sigma(n)$*,* S_n*, Permutations, Subgroups}*

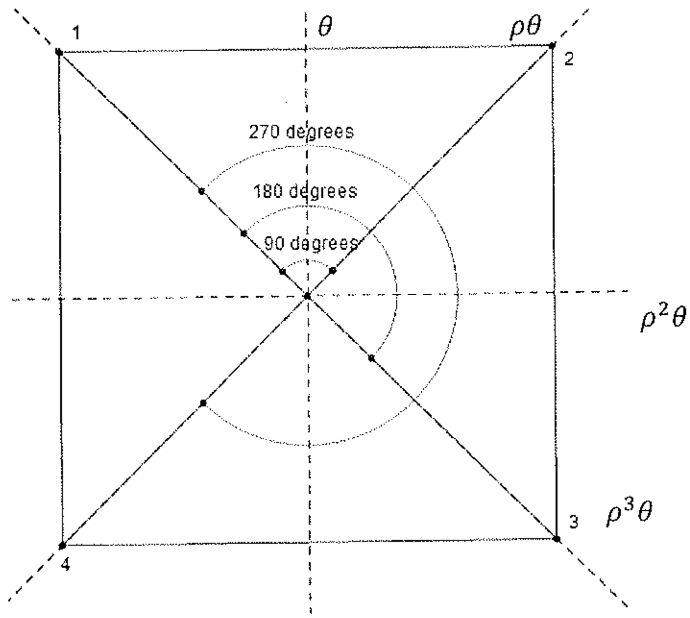
This research paper will discuss the number of subgroups for each dihedral group D(n), $n \geq 3$. The focus in the mathematics of this project is to use basic geometry, group theory and number theory to investigate and develop a formula for the number of subgroups of D(n). In order to visualize the permutations of D(n), I will show first, using basic geometry, that each permutation corresponds to a rigid motion (or combination of rigid motions) of a regular n-sided polygon. Group Theory will be used to investigate/explore the number of subgroups of D(n) for n up to 8. Finally, number theory will be used to develop a formula for the number of subgroups of D(n).

I will begin this paper by focusing upon the background information of D(n). The second section of this paper will focus on the development of a formula that outputs the number of subgroups of D(n) based on the number of sides of a related polygon. The representation of D(3) through D(8) will include a picture of each of the related polygons for the dihedral groups D(n), the composition tables and the lattice of subgroups for D(n). I will include a clear description of how I obtained the formula, proof, and extensions for the number of subgroups of D(n). This paper will conclude with a description of how D(n) can be utilized in a secondary classroom. Since abstract algebra is not a topic primarily focused upon in secondary education, this section will contain a lesson in which students will be asked to determine the symmetries/permutations of various figures. During this lesson students will be introduced to the definition of line of symmetry, rotational symmetry, and composition. Students will be given various figures and asked to list all of the symmetries for each. Next, students will create a system to list the permutations of certain regular polygons. The target learning goal of this lesson is for students to identify the combination of permutations that yield the identity of each figure. The extension of this lesson will be for students to relate the permutations of a two-dimensional figure to a three-dimensional figure.

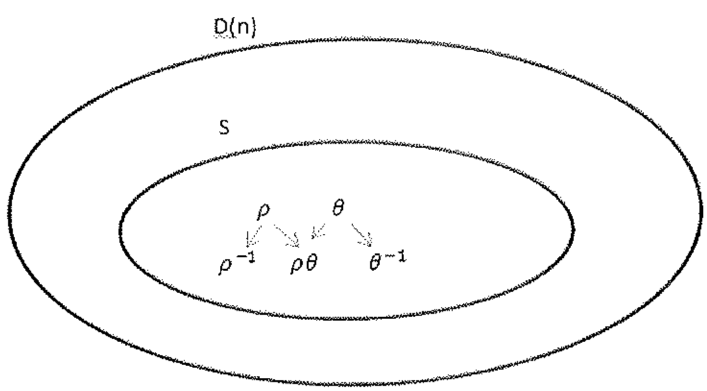
Background Information

In order to better understand the permutations of D(n) it is imperative that we understand what the dihedral groups D(n) are, as well as what they represent. According to Pinter, "For every positive integer n≥3, the regular polygon with n sides has a group of symmetries symbolized by D(n). These groups are called the dihedral groups" (Pinter, 1990). The group of symmetries of a square is symbolized by D(4), and the group of symmetries of a regular pentagon is symbolized by D(5), and so on. In fact, every plane figure that exhibits regularities, also contain a group of symmetries (Pinter, 1990). The groups of symmetries are defined by permutations which preserve distance between every two points. (Bhattacharya, Jain, & Nagpaul, 1994). By definition, "The group of symmetries of a regular polygon P_n of n sides is called the dihedral group of degree n and denoted by D(n)" (Bhattacharya, Jain, & Nagpaul, 1994).

This project will make use of the definition that all of the permutations for each of the dihedral groups D(n) preserve the cyclic order of the vertices of each regular n-gon. This demonstrates the relationship between the abstract concept of D(n) and the rigid motions of a regular n-gon. From this, I will denote the rotation symmetry for each of the regular n-gons of $\frac{2\pi}{n}$ (clockwise) as ρ, the reflection symmetry as θ, and the original n-gon as ε. For example, in the following diagram the identity of the square (ε) is the square in which each vertex and side is matched. Since a square has 4 sides, the rotation ρ is equal to $\frac{2\pi}{4}$ which is a 90^0 clockwise rotation, ρ^2 is equivalent to a 180^0 clockwise rotation, ρ^3 is equivalent to a 270^0 clockwise rotation, and ρ^4 is equivalent to a 360^0 clockwise rotation which is equivalent to the identity ε. θ can be noted as the reflection about the vertical line which passes through the center of the square (as seen below), $\rho\theta$ can be noted as the line of symmetry which passes through the vertices 2 and 4, $\rho^2\theta$ can be noted as the horizontal line of symmetry which passes through the center of the square, and $\rho^3\theta$ can be noted as the line of symmetry which passes through the vertices 1 and 3.



In order to identify all of the subgroups of the dihedral group D(n) it is essential to understand the definition of a subgroup. According to Pinter, a subgroup is defined by, "Let D(n) be a group, and S be a nonempty subset of D(n). If the operation of every pair of elements of S is contained in S, we say that S is closed with respect to that operation. Also, if the inverse of every element of S is in S, then we say that S is closed with respect to inverses. If both of these are true then we call S a subgroup of D(n)" (Pinter, 1990).



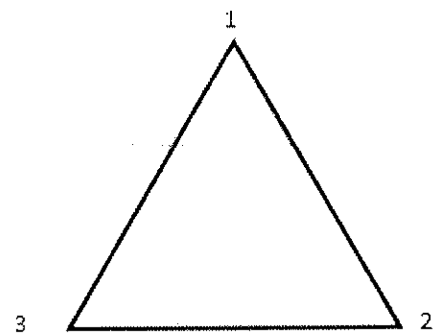
The Dihedral Group D(n)

I will begin this section by describing what each dihedral group D(n) represents. I will define the notation used to create each group D(n), the operation tables, and the lattice of the subgroups for each n. From this, the formula that outputs the number of subgroups of the dihedral group D(n) will be conjectured.

As described in the background information, the two types of symmetries that regular polygons have are rotational symmetry and line symmetry. Each of the rotational symmetries will be labeled as the powers of ρ, each line of symmetry will be labeled as the powers of ρ times θ, and ε will represent the original regular polygon such that the vertices are in their original circular order. Each of the dihedral groups will be represented in this paper using this notation.

To determine the number of subgroups of D(n) and the process to derive the formula I will identify representations for each dihedral group D(n). This includes a) a picture of the regular polygons, b) the elements contained in the group D(n), c) the operation table, and d) the lattice of the subgroups for each D(n). The operation tables define all of the symmetry operations. The operation tables are used to identify the closure of a set; subsequently, identifying all of the subgroups. The lattices of the subgroups begin with the entire group D(n) and will branch to each subgroup that is a subset (or contained) in the group connected above until it reaches the identity ε. The representation for a clockwise rotation of a regular triangle, ρ, will be represented as $\binom{123}{231}$ or 1→2, 2→3 and 3→1. The aforementioned notation will be used for each dihedral group D(n).

D(3)

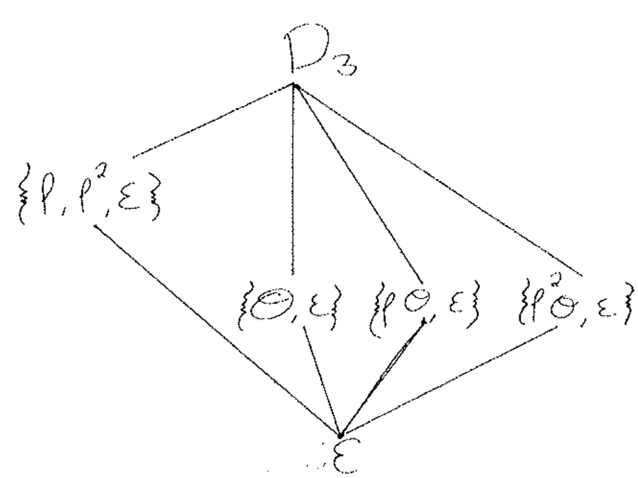


a)

b) D(3)={$\varepsilon = \binom{123}{123}, \rho = \binom{123}{231}, \rho^2 = \binom{123}{312}, \theta = \binom{123}{132}, \rho\theta = \binom{123}{213}, \rho^2\theta = \binom{123}{321}$}

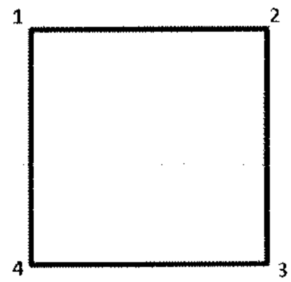
	ε	ρ	ρ^2	θ	$\rho\theta$	$\rho^2\theta$
ε	ε	ρ	ρ^2	θ	$\rho\theta$	$\rho^2\theta$
ρ	ρ	ρ^2	ε	$\rho\theta$	$\rho^2\theta$	θ
ρ^2	ρ^2	ε	ρ	$\rho^2\theta$	θ	$\rho\theta$
θ	θ	$\rho^2\theta$	$\rho\theta$	ε	ρ^2	ρ
$\rho\theta$	$\rho\theta$	θ	$\rho^2\theta$	ρ	ε	ρ^2
$\rho^2\theta$	$\rho^2\theta$	$\rho\theta$	θ	ρ^2	ρ	ε

c)



d)

D(4)



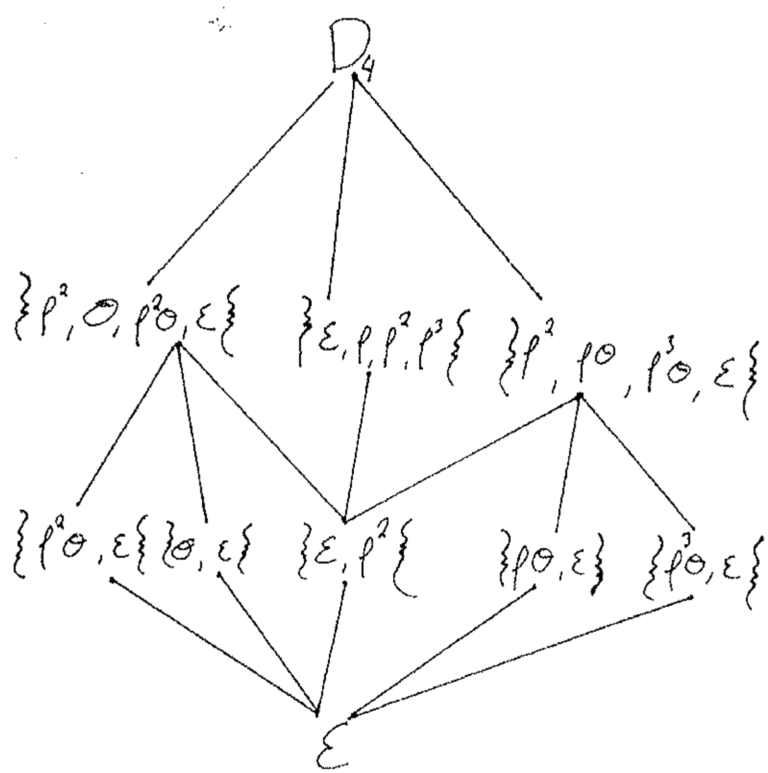
a)

b) D(4)={$\varepsilon = \binom{1234}{1234}, \rho = \binom{1234}{2341}, \rho^2 = \binom{1234}{3412}, \rho^3 = \binom{1234}{4123},$

$$\theta = \binom{1234}{2143}, \rho\theta = \binom{1234}{3214}, \rho^2\theta = \binom{1234}{4321}, \rho^3\theta = \binom{1234}{1432}\}$$

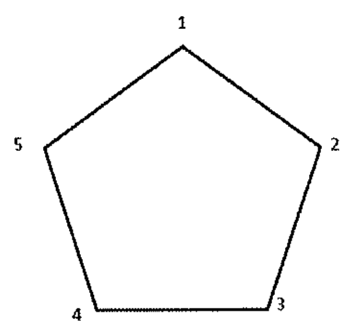
c)

	ε	ρ	ρ^2	ρ^3	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$
ε	ε	ρ	ρ^2	ρ^3	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$
ρ	ρ	ρ^2	ρ^3	ε	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	θ
ρ^2	ρ^2	ρ^3	ε	ρ	$\rho^2\theta$	$\rho^3\theta$	θ	$\rho\theta$
ρ^3	ρ^3	ε	ρ	ρ^2	$\rho^3\theta$	θ	$\rho\theta$	$\rho^2\theta$
θ	θ	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	ε	ρ^3	ρ^2	ρ
$\rho\theta$	$\rho\theta$	θ	$\rho^3\theta$	$\rho^2\theta$	ρ	ε	ρ^3	ρ^2
$\rho^2\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^3\theta$	ρ^2	ρ	ε	ρ^3
$\rho^3\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	ρ^3	ρ^2	ρ	ε



d)

D(5)



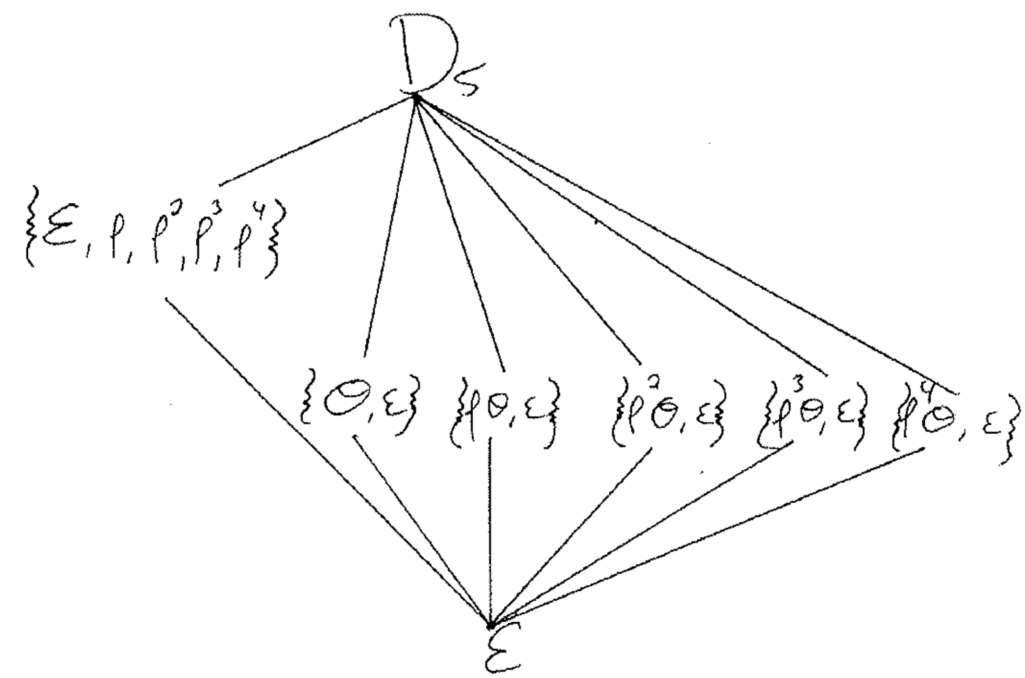
a)

b) D(5)={$\varepsilon = \begin{pmatrix} 12345 \\ 12345 \end{pmatrix}, \rho = \begin{pmatrix} 12345 \\ 23451 \end{pmatrix}, \rho^2 = \begin{pmatrix} 12345 \\ 34512 \end{pmatrix}, \rho^3 = \begin{pmatrix} 12345 \\ 45123 \end{pmatrix}, \rho^4 = \begin{pmatrix} 12345 \\ 51234 \end{pmatrix}$

$\theta = \begin{pmatrix} 12345 \\ 15432 \end{pmatrix}, \rho\theta = \begin{pmatrix} 12345 \\ 21543 \end{pmatrix}, \rho^2\theta = \begin{pmatrix} 12345 \\ 32154 \end{pmatrix}, \rho^3\theta = \begin{pmatrix} 12345 \\ 43215 \end{pmatrix}, \rho^4\theta = \begin{pmatrix} 12345 \\ 54321 \end{pmatrix}$}

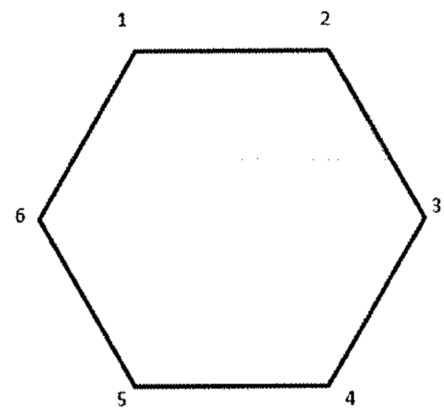
c)

	ε	ρ	ρ^2	ρ^3	ρ^4	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$
ε	ε	ρ	ρ^2	ρ^3	ρ^4	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$
ρ	ρ	ρ^2	ρ^3	ρ^4	ε	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	θ
ρ^2	ρ^2	ρ^3	ρ^4	ε	ρ	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	θ	$\rho\theta$
ρ^3	ρ^3	ρ^4	ε	ρ	ρ^2	$\rho^3\theta$	$\rho^4\theta$	θ	$\rho\theta$	$\rho^2\theta$
ρ^4	ρ^4	ε	ρ	ρ^2	ρ^3	$\rho^4\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$
θ	θ	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	ε	ρ^4	ρ^3	ρ^2	ρ
$\rho\theta$	$\rho\theta$	θ	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	ρ	ε	ρ^4	ρ^3	ρ^2
$\rho^2\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^4\theta$	$\rho^3\theta$	ρ^2	ρ	ε	ρ^4	ρ^3
$\rho^3\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^4\theta$	ρ^3	ρ^2	ρ	ε	ρ^4
$\rho^4\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	ρ^4	ρ^3	ρ^2	ρ	ε



d)

D(6)



a)

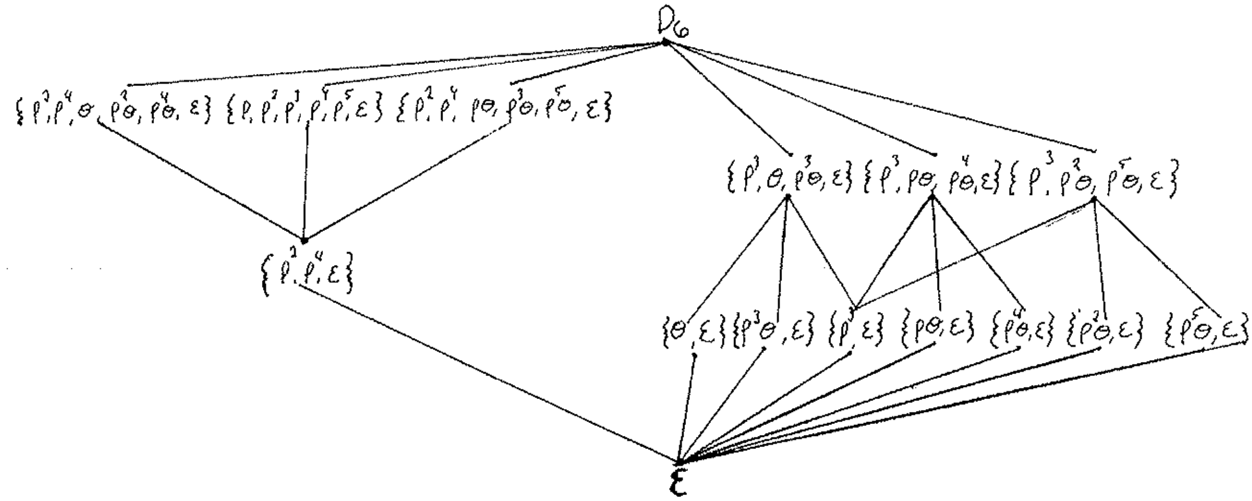
b) D(6)={$\varepsilon = \binom{123456}{123456}, \rho = \binom{123456}{234561}, \rho^2 = \binom{123456}{345612}, \rho^3 = \binom{123456}{456123}, \rho^4 = \binom{123456}{561234},$

$$\rho^5 = \binom{123456}{612345}, \theta = \binom{123456}{165432}, \rho\theta = \binom{123456}{216543}, \rho^2\theta = \binom{123456}{321654}, \rho^3\theta = \binom{123456}{432165},$$

$$\rho^4\theta = \binom{123456}{543216}, \rho^5\theta = \binom{123456}{654321}\}$$

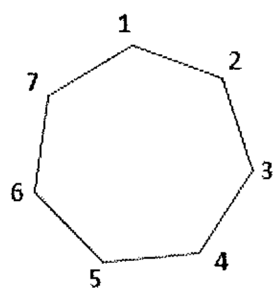
	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$
ε	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$
ρ	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ε	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	θ
ρ^2	ρ^2	ρ^3	ρ^4	ρ^5	ε	ρ	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	θ	$\rho\theta$
ρ^3	ρ^3	ρ^4	ρ^5	ε	ρ	ρ^2	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	θ	ρ	$\rho^2\theta$
ρ^4	ρ^4	ρ^5	ε	ρ	ρ^2	ρ^3	$\rho^4\theta$	$\rho^5\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$
ρ^5	ρ^5	ε	ρ	ρ^2	ρ^5	ρ^4	$\rho^5\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$
θ	θ	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	ε	ρ^5	ρ^4	ρ^3	ρ^2	ρ
$\rho\theta$	$\rho\theta$	θ	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	ρ	ε	ρ^5	ρ^4	ρ^3	ρ^2
$\rho^2\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	ρ^2	ρ	ε	ρ^5	ρ^4	ρ^3
$\rho^3\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^5\theta$	$\rho^4\theta$	ρ^3	ρ^2	ρ	ε	ρ^5	ρ^4
$\rho^4\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^5\theta$	ρ^4	ρ^3	ρ^2	ρ	ε	ρ^5
$\rho^5\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	ρ^5	ρ^4	ρ^3	ρ^2	ρ	ε

c)



d)

D(7)



a)

b) D(7)= $\{\varepsilon = \binom{1234567}{1234567}, \rho = \binom{1234567}{2345671}, \rho^2 = \binom{1234567}{3456712}, \rho^3 = \binom{1234567}{4567123},$

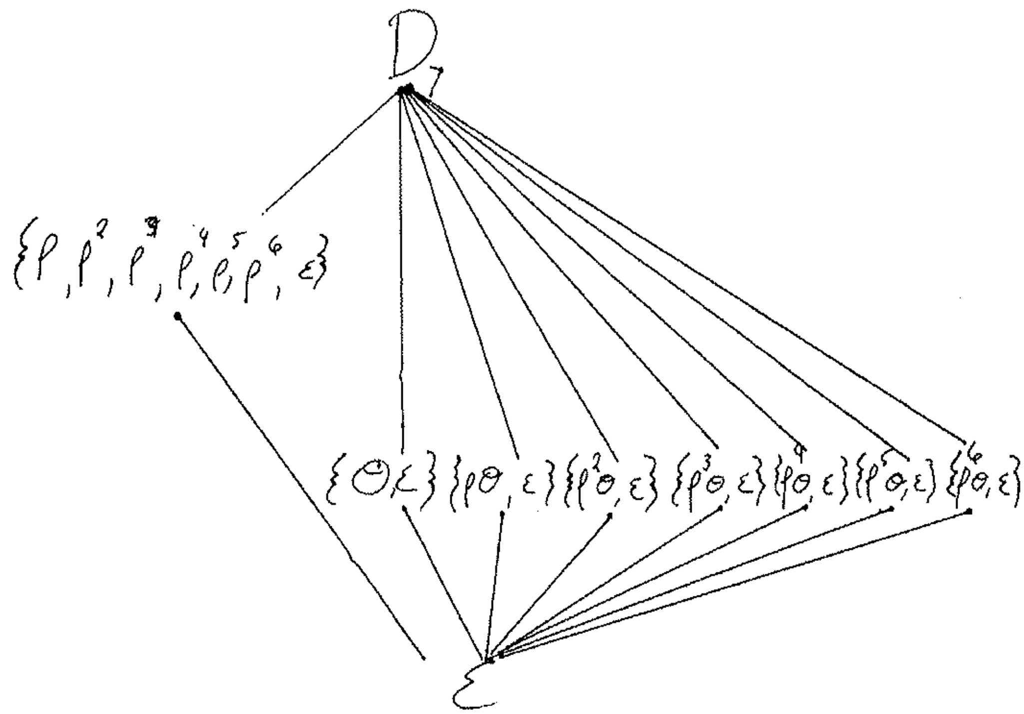
$$\rho^4 = \binom{1234567}{5671234}, \rho^5 = \binom{1234567}{6712345}, \rho^6 = \binom{1234567}{7123456},$$

$$\theta = \binom{1234567}{1765432}, \rho\theta = \binom{1234567}{2176543}, \rho^2\theta = \binom{1234567}{3217654}, \rho^3\theta = \binom{1234567}{4321765},$$

$$\rho^4\theta = \binom{1234567}{5432176}, \rho^5\theta = \binom{1234567}{6543217}, \rho^6\theta = \binom{1234567}{7654321}\}$$

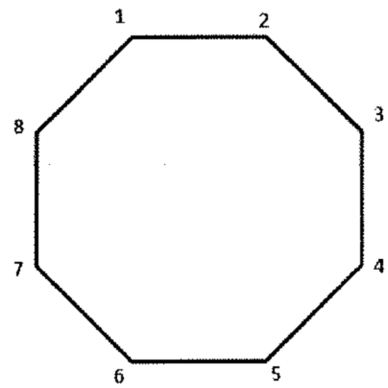
	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$
ε	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$
ρ	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ε	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	θ
ρ^2	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ε	ρ	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	θ	$\rho\theta$
ρ^3	ρ^3	ρ^4	ρ^5	ρ^6	ε	ρ	ρ^2	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	θ	$\rho\theta$	$\rho^2\theta$
ρ^4	ρ^4	ρ^5	ρ^6	ε	ρ	ρ^2	ρ^3	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$
ρ^5	ρ^5	ρ^6	ε	ρ	ρ^2	ρ^3	ρ^4	$\rho^5\theta$	$\rho^6\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$
ρ^6	ρ^6	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	$\rho^6\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$
θ	θ	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	ε	ρ^6	ρ^5	ρ^4	ρ^3	ρ^2	ρ
$\rho\theta$	$\rho\theta$	θ	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	ρ	ε	ρ^6	ρ^5	ρ^4	ρ^3	ρ^2
$\rho^2\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	ρ^2	ρ	ε	ρ^6	ρ^5	ρ^4	ρ^3
$\rho^3\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	ρ^3	ρ^2	ρ	ε	ρ^6	ρ^5	ρ^4
$\rho^4\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^6\theta$	$\rho^5\theta$	ρ^4	ρ^3	ρ^2	ρ	ε	ρ^6	ρ^5
$\rho^5\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^6\theta$	ρ^5	ρ^4	ρ^3	ρ^2	ρ	ε	ρ^6
$\rho^6\theta$	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	ρ^6	ρ^5	ρ^4	ρ^3	ρ^2	ρ	ε

c)



d)

D(8)



a)

b) D(8)={$\varepsilon = \binom{12345678}{12345678}, \rho = \binom{12345678}{23456781}, \rho^2 = \binom{12345678}{34567812}, \rho^3 = \binom{12345678}{45678123},$

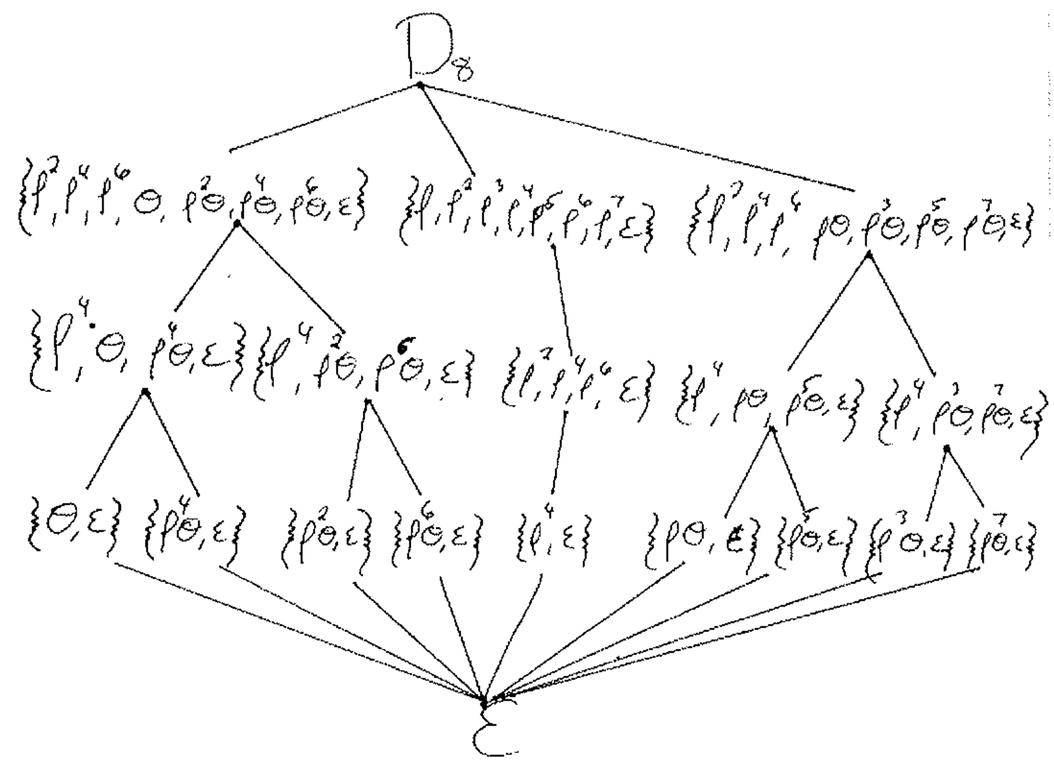
$\rho^4 = \binom{12345678}{56781234}, \rho^5 = \binom{12345678}{67812345}, \rho^6 = \binom{12345678}{78123456}, \rho^7 = \binom{12345678}{81234567}$

$\theta = \binom{12345678}{18765432}, \rho\theta = \binom{12345678}{21876543}, \rho^2\theta = \binom{12345678}{32187654}, \rho^3\theta = \binom{12345678}{43218765},$

$\rho^4\theta = \binom{12345678}{54321876}, \rho^5\theta = \binom{12345678}{65432187}, \rho^6\theta = \binom{12345678}{76543218}, \rho^7\theta = \binom{12345678}{87654321}$}

	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	$\rho^7\theta$
ε	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	$\rho^7\theta$
ρ	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ε	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	$\rho^7\theta$	θ
ρ^2	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ε	ρ	ρ^2	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	$\rho^7\theta$	θ	$\rho\theta$
ρ^3	ρ^3	ρ^4	ρ^5	ρ^6	ρ^7	ε	ρ	ρ^2	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	$\rho^7\theta$	θ	$\rho\theta$	$\rho^2\theta$
ρ^4	ρ^4	ρ^5	ρ^6	ρ^7	ε	ρ	ρ^2	ρ^3	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$	$\rho^7\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$
ρ^5	ρ^5	ρ^6	ρ^7	ε	ρ	ρ^2	ρ^3	ρ^4	$\rho^5\theta$	$\rho^6\theta$	$\rho^7\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$
ρ^6	ρ^6	ρ^7	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	$\rho^7\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$
ρ^7	ρ^7	ε	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^6	$\rho^7\theta$	θ	$\rho\theta$	$\rho^2\theta$	$\rho^3\theta$	$\rho^4\theta$	$\rho^5\theta$	$\rho^6\theta$
θ	θ	$\rho^7\theta$	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	ε	ρ^7	ρ^6	ρ^5	ρ^4	ρ^3	ρ^2	ρ
$\rho\theta$	$\rho\theta$	θ	$\rho^7\theta$	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	ρ	ε	ρ^7	ρ^6	ρ^5	ρ^4	ρ^3	ρ^2
$\rho^2\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^7\theta$	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	ρ^2	ρ	ε	ρ^7	ρ^6	ρ^5	ρ^4	ρ^3
$\rho^3\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^7\theta$	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	ρ^3	ρ^2	ρ	ε	ρ^7	ρ^6	ρ^5	ρ^4
$\rho^4\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^7\theta$	$\rho^6\theta$	$\rho^5\theta$	ρ^4	ρ^3	ρ^2	ρ	ε	ρ^7	ρ^6	ρ^5
$\rho^5\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	$\rho^7\theta$	$\rho^6\theta$	ρ^5	ρ^4	ρ^3	ρ^2	ρ	ε	ρ^7	ρ^6
$\rho^6\theta$	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	ρ^7	ρ^6	ρ^5	ρ^4	ρ^3	ρ^2	ρ	ε	ρ^7
$\rho^7\theta$	$\rho^7\theta$	$\rho^6\theta$	$\rho^5\theta$	$\rho^4\theta$	$\rho^3\theta$	$\rho^2\theta$	$\rho\theta$	θ	ρ^7	ρ^6	ρ^5	ρ^4	ρ^3	ρ^2	ρ	ε

c)



d)

The Collection of the Number of Subgroups of Dihedral Group D(n)

n	Number of Subgroups of D(n)
3	6
4	10
5	8
6	16
7	10
8	19

Now that D(3) through D(8), along with their subgroups have been described, I will now investigate the mathematics behind creating a formula that outputs the number of subgroups for D(n). The formula was contrived through trial and error while I was trying to generate the list of subgroups of D(n). I quickly noted that D(n) will always contain the subgroup D(n), the subgroup ε, and the subgroups generated by ρ^{a_i} where a_i are the positive divisors of n. If k is relatively prime to n then no additional subgroups can be generated by ρ^k. Modular arithmetic demonstrates that a relatively prime number will generate every number contained in the set created by mod(n); therefore, each subgroup corresponds to a factor of n.

I will investigate the subgroups for D(4) and D(8). It is noted for D(4) that the factors of 4 are 1,2, and 4. The subgroups of D(4) are as follows;

$\{D_4\}, \{\rho, \rho^2, \rho^3, \varepsilon\}, \{\rho^2, \varepsilon\}, \{\varepsilon\}, \{\rho^2, \theta, \rho^2\theta, \varepsilon\}, \{\rho^2, \rho\theta, \rho^3\theta, \varepsilon\}, \{\theta, \varepsilon\}, \{\rho\theta, \varepsilon\}, \{\rho^2\theta, \varepsilon\}, \{\rho^3\theta, \varepsilon\}$. I will break these subgroups into two groups: a) subgroups that only contain rotations and b) subgroups that contain reflections.

a) Looking at the three subgroups which contain rotations of the square; ρ will generate the subgroup only containing rotations generated by a 90^0 clockwise rotation, ρ^2 will generate the subgroup of rotations generated by a 180^0 clockwise rotation, and ρ^4 (or ε) will generate the last subgroup that is generated by a 360^0 clockwise rotation. Thus I can conjecture that the number of subgroups of D(4) that only contain rotations is equivalent to the number of factors of 4.

b) I will now investigate the subgroups that contain rotations and reflections. The subgroup generated by ρ and θ will produce the the entire group D(n). The subgroup generated by ρ^2 and θ will produce $\{\rho^2, \theta, \rho^2\theta, \varepsilon\}$. The subgroup generated by ρ^2 *and* $\rho\theta$ will produce $\{\rho^2, \rho\theta, \rho^3\theta, \varepsilon\}$. The subgroups generated by ρ^4 or ε and each individual reflection are $\{\theta, \varepsilon\}, \{\rho\theta, \varepsilon\}, \{\rho^2\theta, \varepsilon\}, and\ \{\rho^3\theta, \varepsilon\}$. All things considered, I am able to conjecture that the number of subgroups of D(4) is equivalent to 3+1+2+4. This is equivalent to the number of factors of 4 plus each factor of 4.

Now that I have investigated the number of subgroups for D(4), D(8) will be explored where there are a total of 19 subgroups. Throughout this section I will examine two categories of subgroups: a) subgroups that only contain rotations and b) subgroups that contain reflections. Identifying how each subgroup of D(8) is generated will reveal the formula that outputs the number of subgroups of D(8).

a) In D(8) the subgroups that only contain rotations are $\{\rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6, \rho^7, \varepsilon\}, \{\rho^2, \rho^4, \rho^6, \varepsilon\}, \{\rho^4, \varepsilon\}\ and\ \{\varepsilon\}$. The subgroup $\{\rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6, \rho^7, \varepsilon\}$ represents the subgroup of rotations generated by ρ, $\{\rho^2, \rho^4, \rho^6, \varepsilon\}$ is the subgroup of rotations that is generated by ρ^2, $\{\rho^4, \varepsilon\}$ is the subgroup of rotations that is generated by ρ^4, and $\{\varepsilon\}$ is the subgroup that is generated by ρ^8 or the identity. The importance of this section is to realize that D(8) has four subgroups that only contain rotations. Notice that eight has four factors of 1, 2, 4 and 8.

b) The subgroups that contain both rotations and reflections are $\{D_8\}, \{\rho^2, \rho^4, \rho^6, \theta, \rho^2\theta, \rho^4\theta, \rho^6\theta, \varepsilon\}, \{\rho^2, \rho^4, \rho^6, \rho\theta, \rho^3\theta, \rho^5\theta, \rho^7\theta, \varepsilon\}, \{\rho^4, \theta, \rho^4\theta, \varepsilon\}, \{\rho^4, \rho\theta, \rho^5\theta, \varepsilon\},$

$\{\rho^4, \rho^2\theta, \rho^6\theta, \varepsilon\}, \{\rho^4, \rho^3\theta, \rho^7\theta, \varepsilon\}, \{\theta, \varepsilon\}, \{\rho\theta, \varepsilon\}, \{\rho^2\theta, \varepsilon\}, \{\rho^3\theta, \varepsilon\}, \{\rho^4\theta, \varepsilon\}, \{\rho^5\theta, \varepsilon\}, \{\rho^6\theta, \varepsilon\}, and\ \{\rho^7\theta, \varepsilon\}$. Similar to the rotations, I will focus on the subgroups along with their reflections that are generated by $\rho, \rho^2, \rho^4, and\ \rho^8$. The subgroup $\{\rho^2, \rho^4, \rho^6, \theta, \rho^2\theta, \rho^4\theta, \rho^6\theta, \varepsilon\}$ is generated by $\rho^2\ and\ \theta$ and $\{\rho^2, \rho^4, \rho^6, \rho\theta, \rho^3\theta, \rho^5\theta, \rho^7\theta, \varepsilon\}$ is the subgroup that is generated by ρ^2 and $\rho\theta$. Notice that ρ^2 generates two subgroups that contain reflections. The subgroup $\{\rho^4, \theta, \rho^4\theta, \varepsilon\}$ is generated by $\rho^4\ and\ \theta$, $\{\rho^4, \rho\theta, \rho^5\theta, \varepsilon\}$ is the subgroup that is generated by $\rho^4\ and\ \rho\theta$, $\{\rho^4, \rho^2\theta, \rho^6\theta, \varepsilon\}$ is the subgroup that is generated by $\rho^4\ and\ \rho^2\theta$, and $\{\rho^4, \rho^3\theta, \rho^7\theta, \varepsilon\}$ is the subgroup that is generated by $\rho^4\ and\ \rho^3\theta$. Thus, ρ^4 will generate four subgroups that contain reflections. The remaining subgroups that contain reflections and the identity are $\{\theta, \varepsilon\}, \{\rho\theta, \varepsilon\}, \{\rho^2\theta, \varepsilon\}, \{\rho^3\theta, \varepsilon\}, \{\rho^4\theta, \varepsilon\}, \{\rho^5\theta, \varepsilon\}, \{\rho^6\theta, \varepsilon\}, and\ \{\rho^7\theta, \varepsilon\}$. Notice that ρ^8 will generate eight subgroups that contain reflections. This section demonstrates that D(8) is equivalent to 4+1+2+4+8 which results in 19 total subgroups. Therefore, the number of subgroups of D(8) is equal to the number of factors of eight plus each factor of eight.

Throughout this section I will refer to each lattice of D(n)'s subgroups to validate my conjecture S_n will represent the number of subgroups of D(n). The number of subroups for D(3) is represented as S_3. The collection of subgroups of D(n) demonstrates that S_3 is 6 and the factors of 3 are 1 and 3, then S_3 is 2+3+1, or 6 total subgroups. Similarly, D(5) has 8 subgroups, and my conjecture states that $S_5 =$ 2+1+5, or 8 total subgroups. The formula that determines the number of subgroups of D(n) is gleaned from the lattice of subgroups, as is reflected in the table below.

n	Number of Subgroups of D(n)	S_n
3	6	2+3+1
4	10	3+1+2+4
5	8	2+1+5
6	16	4+1+2+3+6

7	10	2+1+7
8	19	4+1+2+4+8

As stated earlier, the symmetries of any regular n-sided polygons are the elements of D(n), and the subgroups of D(n). The function which determines the number of subgroups of D(n) will utilize τ and σ. By definition, "Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of *n*, and $\sigma(n)$ denote the sum of these divisors" (Burton, 1976). For example, the number 14 has the positive divisors 1, 2, 7, and 14 which means $\tau(14)$=4 and $\sigma(14) = 1 + 2 + 7 + 14 = 24$; consequently, S_{14} is equal to four plus twenty four.

The reader's reflection should be, "Does this formula work for every dihedral group D(n)?"

The Fundamental Theorem of Arithmetic states, If n = $p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ is the prime factorization of n > 1, then the positive divisors of n are precisely those integers d of the form d = $p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, where $0 \leq a_i \leq k_i$ (i=1,2,..., r) stands. This implies that if d is a divisor of n, then d will generate the subgroup of rotations $\rho^d, \rho^{2d}, \dots, \varepsilon$, a subgroup of D(n). From this we can determine the number of subgroups of D(n). Let's begin by determining the value for S_{24}. The factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. The prime factorization of 24 is $2^3 3^1$; thus, 24 has (3+1)(1+1) = 8 positive factors, implying that D(24) has eight subgroups that contain only rotations. The sum of the factors is $1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 56 = \sigma(24)$, implying that D(24) has 56 subgroups that contain reflections; therefore, $S_{24} = \tau(24) + \sigma(24) = 8 + 56 = 64$. D(24) has 64 subgroups.

Proof

Now that we have an understanding of how each subgroup is generated, and know that the formula works for S_3 through S_8, I will prove that $S_n = \tau(n) + \sigma(n)$ for any given n. The proof for this consists of two parts: a) prove that $\tau(n)$ represents the number of subgroups that only contain rotations and b) prove that $\sigma(n)$ represents the number of subgroups that contain reflections.

a. By definition, τ(n) denotes the number of positive divisors of a positive integer n. Let d and n represent positive integers such that d is a divisor of n. Since d is a divisor of n then there exists an m such that m=n/d. ρ^d will generate the closed set $\{\rho^d, \rho^{2d}, \rho^{3d}, \dots, \rho^{n-d}, \rho^n\}$ of rotations. In order for this closed set generated by ρ^d to be a subgroup, it must contain the inverse for every element in the set using properties of exponents, $\rho^d \cdot \rho^{n-d} = \rho^n = \varepsilon, \rho^{2d} \cdot \rho^{n-2d} = \varepsilon, \rho^{3d} \cdot \rho^{n-3d} = \varepsilon, \cdots$. Because the set generated by ρ^d is closed and contains the inverse of each element, then the set generated by ρ^d is a subgroup. This demonstrates that every power of ρ which is a divisor of n will generate a subgroup of rotations. Also, any multiple of d that is not also a divisor of n will generate the same subgroup as ρ^d, and any power of ρ that is relatively prime to n will generate the same subgroup as ρ. In conclusion, the number of subgroups of D(n) that only contain rotations is equal to the number of divisors of n which can be symbolized by τ(n).

b. By definition, σ(n) is the sum of the positive divisors of n. I want to prove that σ(n) represents the number of subgroups that contain reflections. Let the variables z, n and d represent positive integers such that d is a divisor of n, σ(n)=z+d, and $0 \le a_i \le d$ (a_i=1,2,..., d). The subgroups generated by ρ^d and $\rho^{a_i}\theta$ can be listed as; $\{\rho^d, \rho^{2d}, \dots, \varepsilon, \theta, \rho^d\theta, \rho^{2d}\theta, \dots\}, \{\rho^d, \rho^{2d}, \dots, \varepsilon, \rho\theta, \rho^{d+1}\theta, \dots\}$, $\{\rho^d, \rho^{2d}, \dots, \varepsilon, \rho^2\theta, \rho^{d+2}\theta, \dots\}, \dots, \{\rho^d, \rho^{2d}, \dots, \varepsilon, \rho^{d-1}\theta, \rho^{2d-1}\theta, \dots\}$. Each subgroup generated by ρ^d and $\rho^{a_i}\theta$ will contain a specific element from the set $\{\theta, \rho\theta, \rho^2\theta, \dots, \rho^{d-1}\theta\}$. This set has a total of d elements which means that each ρ^d and $\rho^{a_i}\theta$ will generate d subgroups that contain reflections; therefore, the number of subgroups of D(n) that contain reflections is equal to the sum of the divisors of n. Since $\tau(n)$ represents the subgroups only containing rotations and $\sigma(n)$ represents the subgroups containing reflections, then $S_n = \tau(n) + \sigma(n)$ for any given n.

Extension of S_n

In this section I will discuss a few properties that seem to arise from the table that represents S_3 through S_{100}. First I will investigate the number of subgroups represented by S_{2^k}. Since n is an integer in the form 2^k, then $\sigma(2^k)$ can be represented as the geometric series $1+2+4+8+16+...+2^k = \frac{1-2^{k+1}}{-1} = 2^{k+1} - 1$, and $\tau(2^k)$ is equal to k+1; therefore, $S_{2^k} = 2^{k+1} + k$. For example, $S_{32}=2^6+5=64+5=69$ can be checked using the table below and works for any n given that n is in the form 2^k.

When n is a prime number it will result in a pattern. Suppose n is some prime number, then $S_n=(1+n)+2=n+3$; hence, $S_{19}=20+2=22$. I conjecture that the difference of two primes is equal to the difference of the number of subgroups for those prime numbers. The proof of this begins by denoting two prime numbers as n_2 and n_1. As stated earlier, $S_{n_2} - S_{n_1}$ is equal to the difference of (n_2+3) and (n_1+3) which simplifies to $n_2 - n_1$; therefore, the difference of two primes is equal to the difference of the number of subgroups for those prime numbers. For example, $S_{59} - S_{17}=62-20=42=59-17$.

Another interesting pattern arises when S_n is odd. The table demonstrates that S_8, S_{18}, S_{32}, S_{50}, S_{72}, and S_{98} are the only times in which S_n is odd in the table below. Because the sequence 8, 18, 32, 50, 72, and 98 can be expressed with the formula $2x^2+4x+2$, where x is a positive integer, then S_{2x^2+4x+2} will result in an odd number. The only time that S_3 through S_{100} will result in a prime number is when n is equal to 8.

The table below demonstrates that there are many dihedral groups that have the same number of subgroups. For example, D(4) and D(7) have ten subgroups. This can be shown using the formula

$S_n = \tau(n) + \sigma(n)$. The number of subgroups of D(4) can be represented as, $S_4 = \tau(4) + \sigma(4) = 3 + 1 + 2 + 4 = 10$, and $S_7 = \tau(7) + \sigma(7) = 2 + 1 + 7 = 10$. Since, $\tau(7) + \sigma(7) = \tau(4) + \sigma(4)$, then S_7 is equal to S_4. The mathematics behind this project could be extended to finding other groups D(n) and D(m) such that S_n and S_m are equal to each other, but I will not pursue that here.

n	τ(n)	σ(n)	S_n
3	2	4	6
4	3	7	10
5	2	6	8
6	4	12	16
7	2	8	10
8	4	15	19
9	3	13	16
10	4	18	22
11	2	12	14
12	6	28	34
13	2	14	16
14	4	24	28
15	4	24	28
16	5	31	36
17	2	18	20
18	6	39	45
19	2	20	22
20	6	42	48
21	4	32	36
22	4	36	40
23	2	24	26
24	8	60	68
25	3	31	34
26	4	42	46
27	4	40	44
28	6	56	62
29	2	30	32
30	8	72	80
31	2	32	34
32	6	63	69
33	4	48	52
34	4	54	58

n	τ(n)	σ(n)	S_n
35	4	48	52
36	9	91	100
37	2	38	40
38	4	60	64
39	4	56	60
40	8	90	98
41	2	42	44
42	8	96	104
43	2	44	46
44	6	84	90
45	6	78	84
46	4	72	76
47	2	48	50
48	10	124	134
49	3	57	60
50	6	93	99
51	4	72	76
52	6	98	104
53	2	54	56
54	8	120	128
55	4	72	76
56	8	120	128
57	4	80	84
58	4	90	94
59	2	60	62
60	12	168	180
61	2	62	64
62	4	96	100
63	6	104	110
64	7	127	134

n	τ(n)	σ(n)	S_n
65	4	84	88
66	8	144	152
67	2	68	70
68	6	126	132
69	4	96	100
70	8	144	152
71	2	72	74
72	12	195	207
73	2	74	76
74	4	114	118
75	6	124	130
76	6	140	146
77	4	96	100
78	8	168	176
79	2	80	82
80	10	186	196
81	5	121	126
82	4	126	130
83	2	84	86
84	12	224	236
85	4	108	112
86	4	132	136
87	4	120	124
88	8	180	188
89	2	90	92
90	12	234	246
91	4	112	116
92	6	168	174
93	4	128	132
94	4	144	148

n	τ(n)	σ(n)	S_n
95	4	120	124
96	12	252	264

n	τ(n)	σ(n)	S_n
97	2	98	100
98	6	171	177

n	τ(n)	σ(n)	S_n
99	6	156	162
100	9	217	226

Bringing the Dihedral Groups D(n) into the Classroom

An interesting aspect of this project is that I take the geometric concepts of the symmetries of regular polygons (all of which are the elements of D(n)) and utilize abstract algebra by investigating the number of subgroups of the dihedral group D(n). The research closes nicely with a formula that utilizes basic number theory properties. Rather than applying abstract algebra and number theory concepts, geometric principles will be applied throughout the innovation of this project. The overarching goal of this project is to allow high school students the opportunity to work with complex mathematical topics that are more commonly found in undergraduate courses.

This present innovation begins with defining line of symmetry, rotational symmetry and the center of symmetry. Students will be asked to determine the number of line symmetries and rotational symmetries in a number of figures. In addition, students will be asked to create a mapping for each of the symmetries and list the combinations of symmetries that result in the identity or original figure. The tasks that students will complete during the activity entail identifying how many lines of symmetry and rotations of symmetry each figure contains, creating a mapping for each of the symmetries and listing the combination of symmetries which result in the identity. This activity also includes an extension in which students identify the planes of symmetry and the rotational symmetries of a three-dimensional figure.

The activity includes leveled learning goals to encourage all students to strive for the chance to achieve a level of success. The target learning goal for this lesson requires students to form conclusions about the combination of symmetries that result in the identity, like ρ, followed by ρ, followed by ρ in

D(3). In addition to the target learning goal there are two simplified learning goals; the first asks students to identify the number of symmetries in each figure, and the second requires students to create a mapping for each of the symmetries. The complex learning goal requires students to investigate the similarities and differences of the three-dimensional and two-dimensional figures.

The target and simplified learning goals will be assessed using an equilateral triangle, a six leaf flower and two other figures; both containing lines of symmetry and rotational symmetries. The complex goal will be assessed using a regular cube. For this assessment, students will replace lines of symmetry with planes of symmetry, and rather than use a point to rotate the figure, students will use a line. Since all of the elements in each of the dihedral groups D(n) contain each of the symmetries of a regular n-sided polygon, this lesson will require students to find all of the symmetries of an equilateral triangle and to extend it to figures which aren't polygons. Although the mathematics behind this project is intended to find the number of subgroups of D(n), the symmetries that each dihedral group D(n) contain can be found using figures which are not regular n-sided polygons.

The lesson plan template is a requirement for teachers to use in Weld County School District 6. Also, this section includes the worksheet is designed to measure a student's ability to reach each learning goal as explained.

Topic: Identifying Symmetries	**Grade/Class:** 10^{th} Grade/ Geometry	**Date:** 3-27-14
Standards: **Experiment with transformations in the plane** HS.G.CO.2 - Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch). HS.G.CO.3 - Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself. Visualize relationships between two-dimensional and three dimensional objects. **Visualize the relationships between two-dimensional and three-dimensional objects** HS.G.GMD.4 - Identify the shapes of two-dimensional cross-sections of three dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.		

(Common Core State Standards Initiative, 2013)

Content Objective(s):

- Target Learning Goal- Students will be able to form conclusions of how many combinations of symmetries will result in the identity.
- Simpler Goal- Students will be able to identify the number of symmetries each figure has.
- Simpler Goal 2- Students will be able to create a mapping for each of the symmetries.
- Complex Learning Goal- Students will investigate what the similar and differing features of three dimensional and two dimensional figures are.

Language Objective(s):

- Students will be able to write a clear conclusion explaining the relationship between the symmetries of a two-dimensional figure and a three-dimensional figure.
- Students will be able to express each symmetry in unity.

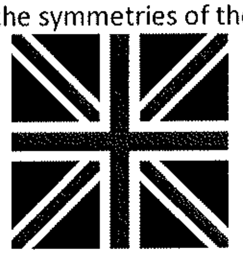
Key Vocabulary:
rotational symmetry, line of symmetry, mapping, and identity

Essential Question(s):
How many symmetries does a cube have?

Time:	Instructional Techniques & Activities: (WICR strategies can be integrated throughout)
5-10 minutes	See below

Introduce/Launch/Engage:

- Warm-Up
  - List or draw all of the symmetries of the flag below



  - Give students 5 minutes
  - Have students discuss their answers. Try to find various students to write up the symmetries.
    - Make a column for reflective symmetry and rotational symmetry
  - Discuss what it means to have all of the symmetries listed. (8 total)
- Discuss the objectives of the day
  - Target Learning Goal- Students will be able to form conclusions of how many combinations of symmetries will result in the identity.
  - Simpler Goal- Students will be able to identify the number of symmetries each figure has. Simpler Goal 2- Students will be able to create a mapping for each of the symmetries.

<table>
<tr>
<td></td>
<td>
<li style="list-style:circle">Complex Learning Goal- Students will be able to investigate what the similar and differing features of three dimensional and two dimensional figures are.
</td>
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<tr>
<td>30-40 minutes</td>
<td>
Teacher/Explore:

Hand out the activity sheet to the class
Have students cut out each figure to use as a manipulative.
Have students label each vertex.
Define rotational symmetry and line of symmetry.

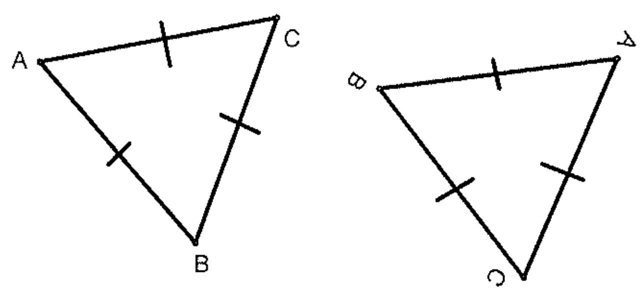
This is included on the student worksheet.
Do a “read aloud” and have students annotate the definitions so that they understand what each symmetry represents.
Tie the definitions to the warm-up if needed

Allow students to explore the symmetries of the equilateral triangle with their group.

As the class finishes the first section lead classroom discourse in which students discuss/agree upon how many rotational symmetries (3) and line of symmetries (3) the triangle has

This uses the fact that the order of D(3) is equal to 6 or 2n, where n is 3
Make the connection and try to formulate why there are 6 symmetries.

Introduce the mapping notation
Ex. A 120⁰ counter clockwise rotation can be represented as a two line matrix $\left(\begin{smallmatrix}abc\\cab\end{smallmatrix}\right)$, where a→c→b→a.



Be sure to identify that the original triangle can be mapped to any symmetry identified that exists.
If a triangle is rotated 120⁰ clockwise, then a→c, b→a and c→b.
Have students create all of the mappings
Have students come to the board to list all of the mappings.

Define Composition

Be sure to explain that we can operate each of the symmetries together.
For example, a 120⁰ rotation followed by a 120⁰ rotation will result in a 240⁰ rotation.

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</table>

	▪ Explain that the goal is to find the list of symmetries which result in the identity in 1, 2, or 3 moves. o Extension question- How many lists of symmetries are there? Are there more? • Repeat for each figure o Note that the worksheet asks students to label those vertices which are necessary. Be sure students only label the vertices which are necessary to identify all of the symmetries. o Have students share their results for each figure. o Ask students if the symmetries of the non-polygon figures share the same symmetries of some n-sided polygon.
5-10 minutes	Close/Summarize/Explain: • Lead classroom discourse about problem number 5 o Why does the question ask for planes of symmetry? o How do the symmetries of a cube relate to the symmetries of a square? o Can you list all of the symmetries? o Create a mapping for all of the symmetries. • Have students conclude how a 2-dimensional object can be related to a 3-dimensional object o Give students objects that represent cubes (die, books, bins, etc.) ▪ Suggest that students use these figures to help guide or check their solutions contained in their writing. o Choose students to discuss their conclusion with the class. o Emphasize that the symmetries and patterns utilized in this lesson can be applied to more than 2-dimensional figures and to other branches of mathematics.

Analysis of Innovation

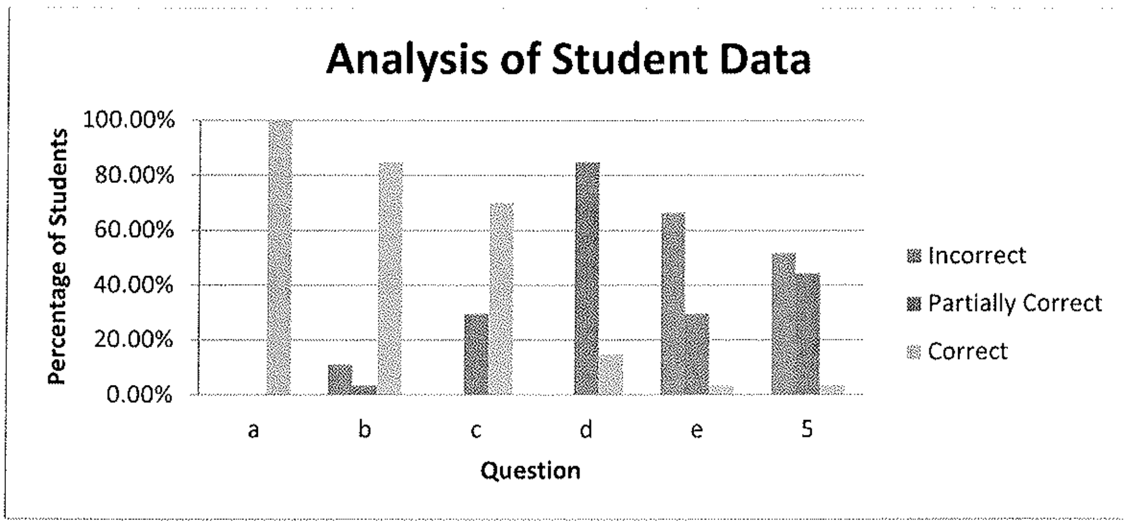
The data collected from this analysis was provided by twenty-seven 10th grade students in Geometry during a 50 minute sophomore-leveled class. Based upon what students had previously learned about line and rotational symmetry, I found that the majority of the class demonstrated success throughout the warm-up and launch of this lesson. Time restraints forced me to use questions two through four as extension problems for students who finished early or would benefit from extra practice. The analysis of questions 1 through 5e will now follow.

The target learning goal for this lesson requires students to form conclusions about how many combinations of symmetries result in the identity. The first simplified learning goal asks students to identify the number of symmetries in each figure. The second simplified learning goal requires students to list the permutations for each of the symmetries. The complex learning goal requires students to investigate the similarities and differences of the three-dimensional and two-dimensional figures. On the first page of the worksheet questions a and b are used to identify which students could meet the first simpler goal, question c is used to identify which students can meet the second simpler goal, questions d and e are used to identify which students can meet the target goal, and question 5e is used to identify which students can meet the more complex goal.

Since this was an obtrusive formative assessment, I chose to use Marzano's method of collecting data to determine the pattern of responses for a particular student (Marzano, Formative Assessment & Standards-Based Grading, 2010). The coding scheme I used for this analysis is: C = totally correct, I = incorrect, and P = partially correct. This coding scheme allowed me to compile the data for the entire class in a manner which was easy to comprehend. An example of my coding scheme for a particular student's work is included below. This student demonstrated that she has met the expectations for the simpler goals, approaching the target goal, and also approaching the more complex goal. This student did not determine all of the combinations of symmetries that would yield the original triangle. She explained very clearly how the reflections of a square translate to a cube, but did not include how the rotations of a square translate to a cube.

Question	Code
A	C
B	C
C	C
D	P
E	I
5e	P

I compiled the entire class's data in Excel so that I could create a bar graph to represent the percentage of students who were able to meet or exceed each learning goal. As identified in the chart below, the entire class determined how many lines of symmetry the triangle contained, in contrast to the eighty-percent of the students that determined the rotational symmetries. Every student who answered question b incorrectly wrote down that the triangle has six rotational symmetries. I found that some of these students double counted rotational symmetries by including both counter and clockwise rotations. The data also demonstrates that nearly seventy-percent of students met the second simpler goal, consequently nearly thirty percent of students approached this learning goal. The students who scored a PC on this question wrote the permutations correctly, but did not include the permutation for each of the symmetries. A little over eighty six-percent of the class answered question d partially correct. The big problem with students' responses for question d is that they were unable to list all symmetries, or explain if there are any more combinations of symmetries that yield the identity. Thinking back to how long it took me to determine the number of subgroups of D(3) through D(8), I understand that this is not something that can be done quickly. With that being said, I conjecture that given more time; the number of students that get questions d and e totally correct would increase. Furthermore, it is my belief that given the opportunity students utilizing manipulatives during this lesson would outperform students without manipulatives. Question 5e provided a range of responses. A little over fifty percent of the class answered this question totally incorrect. Forty-four percent of the students were partially correct on this question. Of the partially correct responses, seven omitted the rotational symmetries, five double counted symmetries, and three demonstrated a misunderstanding of a cube. One student stated that a cube contained four faces that are squares, while others provided general statements of how a cube is composed of square faces, and did not answer the question pertaining to symmetries.



Question	I	PC	C
a	0.00%	0.00%	100.00%
b	11.11%	3.70%	85.19%
c	0.00%	29.63%	70.37%
d	0.00%	85.19%	14.81%
e	66.67%	29.63%	3.70%
5	51.85%	44.44%	3.70%

The intention of this lesson was not to teach students the symmetries of various figures, but to teach students to create and list the combinations of permutations which yield the identity. The lesson was intended to encourage students to go through a process I experienced when finding the subgroups of D(n). Questions d and e inadvertently asked students to go through the same process I did when determining the number of subgroups of D(3) through D(8), consequently it was not reasonable to think that students could achieve this in 50 minutes. Extending this lesson over two to three class periods would increase the number of students that meet the target and more complex learning goals of this lesson. This lesson effectively exposed students to permutations and allowed them to list combinations of permutations that yielded the identity. The result of Question 5e revealed that students were able to extend some of the symmetries of a square to a cube. The majority of the class understood that a line of symmetry of a square will extend to a plane of symmetry with a cube. Students demonstrated that they did not understand that a line of symmetry on a face of the cube will share the same plane of symmetry with the cubes opposing face's line of symmetry. The revisions I have included in the lesson plan is to give each student a cube or die to visualize the symmetries of a cube, give students cut-outs for each figure, and allow students to have more time investigating the combination of permutations that yield the identity. I conjecture that these revisions will improve each student's ability to reach the learning goals associated with this lesson.

Conclusion

The innovation of bringing the dihedral groups D(n) into the classroom allowed me to expose students to the elements of D(n) as well as the process I used to create each subgroup of D(n). The data that I collected from students allowed me to understand that in a 50 minute class period students were able to identify each of the symmetries, list various permutations, and identify some combination of permutations that yield the identity.

In order to determine the number of subgroups for D(3) through D(8), it was important to understand the pattern embedded in the problem. I found that the number of subgroups of D(n) is equal to the number of factors of n plus each positive factor of n. In my classroom I emphasize the value of determining or exploring patterns in mathematics. D(8) allowed me to realize that each subgroup of

D(n) was generated by some ρ^k *and* $\rho^{a_i}\theta$ such that k is a divisor of n and $0 \leq a_i \leq k$. As I have stated throughout this paper, the emphasis of this project is not the formula that determines the number of subgroups of D(n), but rather the connection this project makes through the various fields of mathematics. Each subgroup of D(n) is generated by specific symmetries contained in D(n), the elements of D(n) are the symmetries of the regular n-sided polygon, and the number of subgroups of D(n) is determined by the number of positive divisors of n. This project has demonstrated that a basic geometric property can be approached through group theory and expressed as a formula using number theory. Ultimately, my research has led me to the formula $S_n = \tau(n) + \sigma(n)$.

Student Cut Outs

Cut along the dotted lines.

Equilateral Triangle

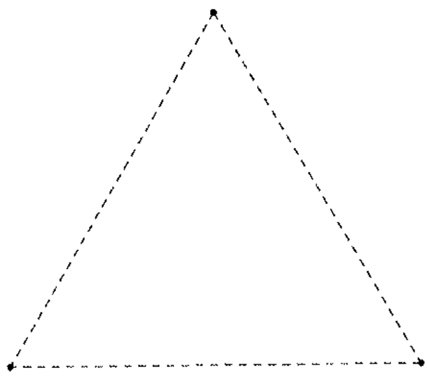


Figure 2

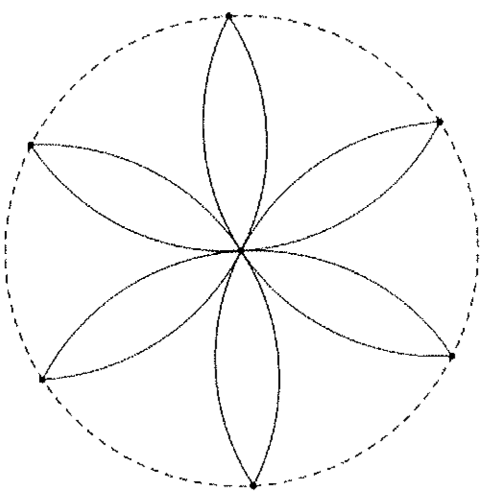


Figure 3

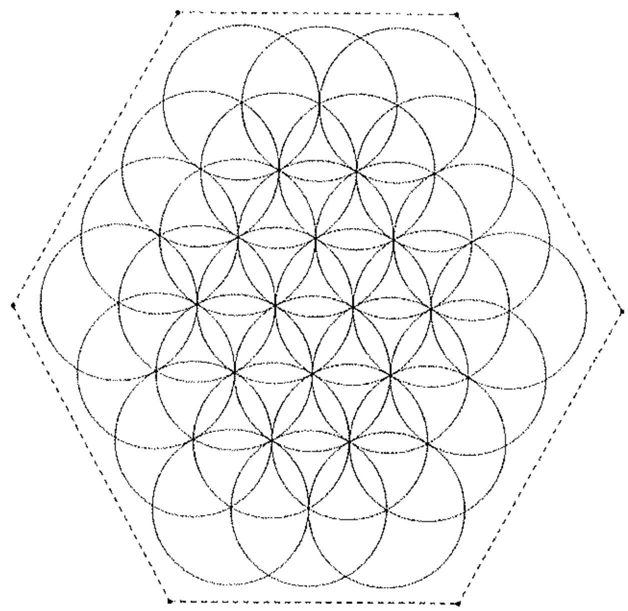
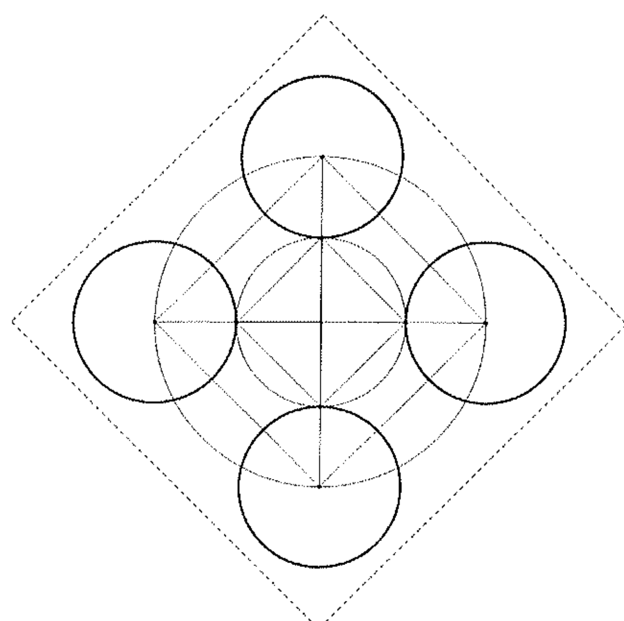


Figure 4



Identifying Symmetries

Definition:

- A figure in the plane has **line symmetry** if the figure can be mapped onto itself by a reflection in a line. This line of reflection is a **line of symmetry**.
- A figure in a plane has **rotational symmetry** if the figure can be mapped onto itself by a rotation of 180^0 or less about the center of the figure. This point is the **center of symmetry.** Note that the rotation can be clockwise or counterclockwise.

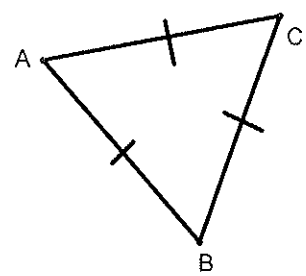
(Larson, Boswell, Kanold, & Stiff, 2007)

- The symbol "•" means composition or "followed by" left to right and every rotation is clockwise

i.e. $120^0 \bullet 120^0 \bullet 120^0 = \binom{abc}{cab}\binom{abc}{cab}\binom{abc}{cab} = \binom{abc}{abc}$, in which a→c→b→a, b→a→c→b, c→b→a→c

1. Below is an equilateral (regular) triangle.
   a. How many lines of symmetry does the triangle contain?

   ____

   b. How many rotational symmetries does the triangle contain?

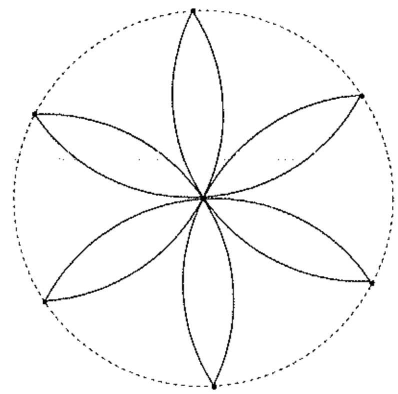
   ____

   c. The permutation (acb) meaning $\binom{abc}{cab}$, corresponds to a clockwise rotation of 120^0 of the triangle above. List each permutation for the symmetries above, using the notation.

   d. List all of the combinations of symmetries that yields the original triangle using:
      i. 1 combination
      ii. 2 combinations
      iii. 3 combinations

   e. Are there any other transformations that return the triangle to its original position without repeating any of the operations above?

2. Use the figure to the right to answer the questions below. Be sure to label each of the symmetries.



   a. How many lines of symmetry does the figure contain?

   ____

   b. How many rotational symmetries does the figure contain?

   ____

   c. List each permutation for the symmetries above, using the notation.

   d. List all of the combinations of symmetries that yields the original triangle using:

      i. 1 combination

      ii. 2 combinations

      iii. 3 combinations

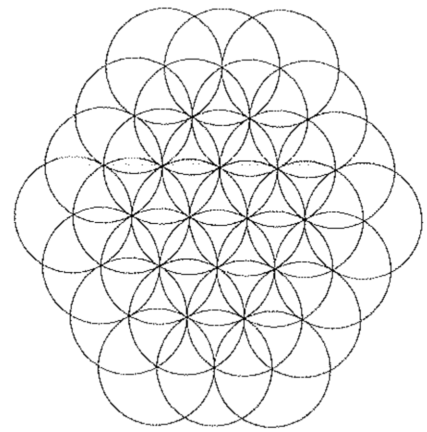
      iv. 4 combinations

      v. 5 combinations

      vi. 6 combinations

   e. Are there any other transformations that return the figure to its original position without repeating any of the operations above?

3. Use the figure to the right to answer the questions below. Be sure to label each of the symmetries.



   a. How many lines of symmetry does the figure contain?

      ____

   b. How many rotational symmetries does the figure contain?

      ____

   c. List each permutation for the symmetries above, using the notation.

   d. List all of the combinations of symmetries that yields the original triangle using:

      i. 1 combination

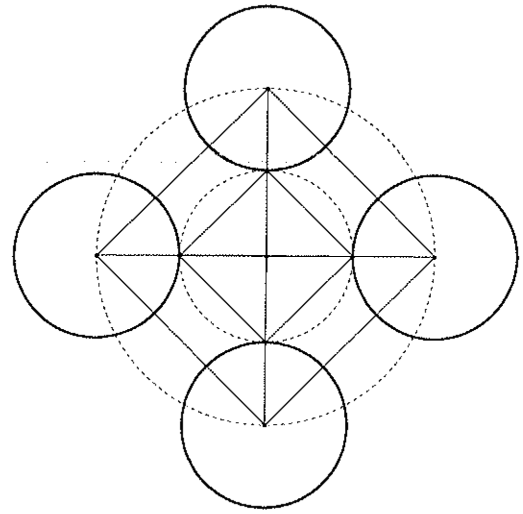
      ii. 2 combinations

      iii. 3 combinations

      iv. 4 combinations

   e. Are there any other transformations that return the figure to its original position without repeating any of the operations above?

4. Use the figure to the right to answer the questions below. Be sure to label each of the symmetries.



   a. How many lines of symmetry does the figure contain?

      ____

   b. How many rotational symmetries does the figure contain?

      ____

   c. List each permutation for the symmetries above, using the notation.

   d. List all of the combinations of symmetries that yields the original triangle using:

      i. 1 combination

      ii. 2 combinations

      iii. 3 combinations

      iv. 4 combinations

   e. Are there any other transformations that return the figure to its original position without repeating any of the operations above?

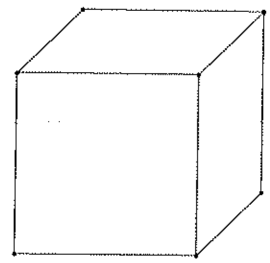
5. Use the regular cube to the right to answer the questions below. Be sure to label each of the symmetries.

   a. How many planes of symmetry does the figure contain?

      ____

   b. How many rotational symmetries does the figure contain?

      ____



   c. List each permutation for the symmetries above.

   d. List all of the combinations of symmetries that yields the original triangle using:

   e. Briefly describe how the symmetries of a 2-dimensional figure can be applied to a 3-dimensional figure. Be sure to refer to the example above.

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